Short Course on Experimental Dynamic Substructuring

Module #4: Simple Exercises

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Short Course Notes For:
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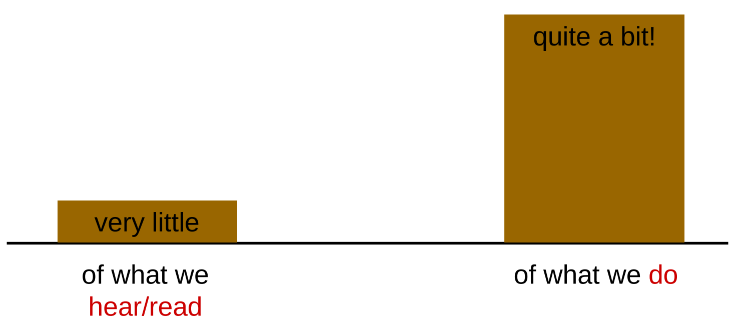
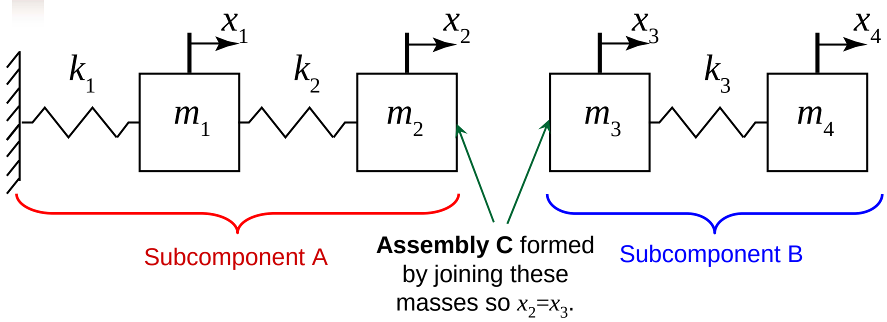
The Science of Learning

- Education research has consistently shown that we retain:



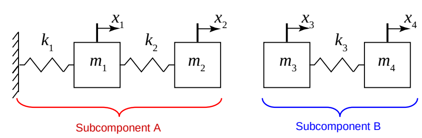
- Our goal in the next 30 minutes is to for you to begin to DO substructuring so the concepts will take root.

Simple Example: discrete mass-spring system



- **Exercises: Compute A+B=C**
  - In the following exercises you will use both FBS and modal substructuring to predict the response/modes of Assembly C using either the FRFs of the substructures or their modes.
  - Exercise #1: FBS
  - Exercise #2: Modal Substructuring

Exercise #1: Discrete Mass Substructuring with FBS



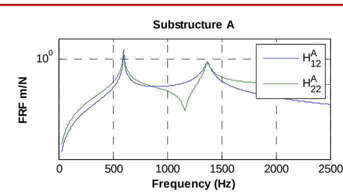
Note: Here we use the IMAC standard modal notation: X=response, H=frequency response matrix (FRF). In Module 2 we used u=response, Y=FRF.

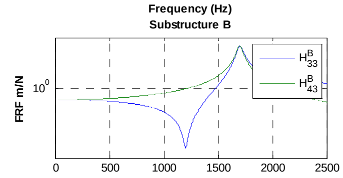
- **Exercise #1:**
  - **Given**: Measurements of $H_{1,2}^A$, $H_{2,2}^A$, $H_{3,3}^B$, $H_{4,3}^B$
  - **Compute** $H_{1,4}^C$ using Frequency Based Substructuring (FBS)
    - Matlab file below includes the simulated measurements shown to the right.
    - The relevant equations are repeated on the next slide.
- **Extra credit**: Starting with:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^A = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix}^A \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^A$$

and similarly for "B", …

  - Derive any expressions that you need by enforcing compatibility (equivalent displacements at the interface) and equilibrium (sum of forces at the interface).

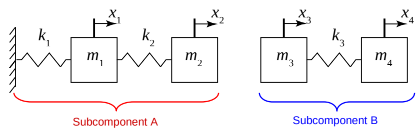


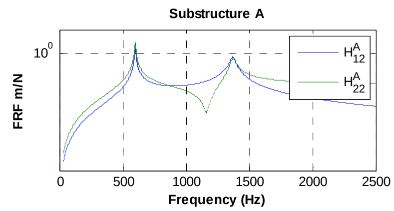


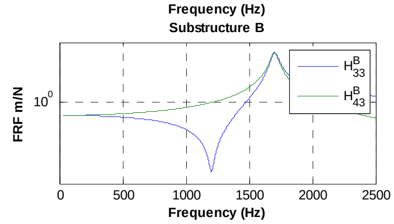
Discrete_Mass_Example.m

Exercise #1- HINTS

- **Exercise #1 (and #2):**
  - Write the EOM for each substructure.
    - In the frequency domain, e.g. **X**(ω)=**H**(ω)**F**(ω) for Exercise #1
    - (In the time domain for #2)
  - Create a concatenated DOF vector that contains $\{x_1, x_2, x_3, x_4\}$
  - Calculate the B matrix that enforces compatibility $x_3=x_4$
  - Calculate L from B and check that equilibrium is satisfied.
  - Use the equations presented earlier, and summarized on the following slide, to compute the FRFs of the assembly.







Review of Frequency Based Substructuring Equations

- Constraints:

$$\boldsymbol{Bu} = \mathbf{0}$$

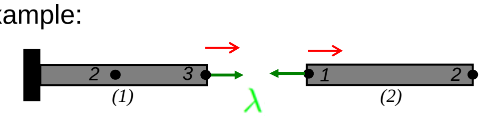
signed Boolean matrix, e.g. [0,1,-1]^T

Note, this copied from the notes earlier and so the notation and node numbering doesn't match that used in these exercises.

- Response of assembly:

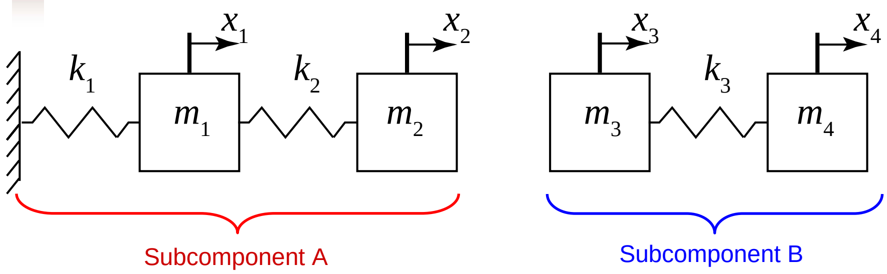
$$\boxed{\boldsymbol{u} = \left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^{\boldsymbol{T}}(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{\boldsymbol{T}})^{-1}\boldsymbol{B}\boldsymbol{Y}\right)\boldsymbol{f}}$$

  - For Example:



$$\boldsymbol{u} = \begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix}$$

Exercise #2: Discrete Mass Substructuring with CMS



- **Exercise #2: A+B=C Using Modal Substructuring**
  - **Given: "**Measured" natural frequencies, ω_r, damping ratios, ζ_r, and mode shapes, ϕ_r. for each subcomponent,
    - Modal parameters are computed in the script below and stored in the variables:
    - `fn_A, zt_A, Phi_A`, etc…
  - **Compute** the natural frequencies and mode shapes of the assembly using Modal Substructuring with **Primal Assembly**.

Discrete_Mass_Example.m

Exercise #2 - HINTS

- **Exercise #2: A+B=C Using Modal Substructuring**
  - **First see the hints for Exercise #1**
  - Start with the subsystem mass, stiffness and damping matrices in modal coordinates (e.g. assuming mass normalized modes).
  - Calculate the modal **B** matrix:

$$\mathbf{B}\begin{Bmatrix}\mathbf{x}^{A}\\ \mathbf{x}^{B}\end{Bmatrix}=\mathbf{0} \qquad \mathbf{B}_{\text{modal}}=\mathbf{B}\begin{bmatrix}\boldsymbol{\Phi}^{A} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Phi}^{B}\end{bmatrix} \qquad \mathbf{B}_{\text{modal}}\begin{Bmatrix}\mathbf{q}^{A}\\ \mathbf{q}^{B}\end{Bmatrix}=\mathbf{0}$$

  - Find the **L** matrix corresponding to $\mathbf{B}_{\text{modal}}$.
  - Use **L** to compute the assembled system matrices, as in the following (from Module 2).

$$L^T\left(ML\ddot{u}_{glob}+CL\dot{u}_{glob}+KLu_{glob}=f+\cancel{g}\right)$$

$$\boxed{\tilde{M}\ddot{u}_{glob}+\tilde{C}\dot{u}_{glob}+\tilde{K}u_{glob}=\tilde{f}}$$

Primally assembled (like FE assembly)

$$\left\{\begin{array}{lcl}\tilde{M} & \triangleq & L^TML\\ \tilde{C} & \triangleq & L^TCL\\ \tilde{K} & \triangleq & L^TKL\\ \tilde{f} & \triangleq & L^Tf\end{array}\right.$$

Additional Exercises:

- **Exercise #3:** Frequency Based Substructuring
  - Add a small amount of random noise to your FRFs and repeat the computation. How sensitive is the solution to errors in your FRFs? Is it particularly sensitive at certain frequencies?
- **Exercise #4:** Modal Substructuring
  - Check how your results vary if you exclude the elastic mode from your solution. Do the results make sense?
  - Add random errors to your identified natural frequencies, damping ratios and mode shapes. How sensitive is the substructuring solution to noise in the modal parameters?
- Once you are done, you can see Allen's solution in: "Discrete_Mass_Example_Solution.m"

Comments on Modal Substructuring Implementation

- Consider the EOM for two independent structures:

$$\begin{bmatrix} \mathbf{I}^A & 0 \\ 0 & \mathbf{I}^B \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}^A \\ \ddot{\mathbf{q}}^B \end{Bmatrix} + \begin{bmatrix} \left[\setminus \omega_r^2 \setminus \right]^A & \mathbf{0} \\ \mathbf{0} & \left[\setminus \omega_r^2 \setminus \right]^B \end{bmatrix} \begin{Bmatrix} \mathbf{q}^A \\ \mathbf{q}^B \end{Bmatrix} = \begin{bmatrix} (\mathbf{\Phi}^A)^T & \mathbf{0} \\ \mathbf{0} & (\mathbf{\Phi}^B)^T \end{bmatrix} \begin{Bmatrix} \mathbf{f}^A \\ \mathbf{f}^B \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbf{u}^A \\ \mathbf{u}_B \end{Bmatrix} = \begin{bmatrix} \mathbf{\Phi}^A & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_B \end{bmatrix} \begin{Bmatrix} \mathbf{q}^A \\ \mathbf{q}_B \end{Bmatrix}$$

- Recall that we connect them by enforcing some constraints:

$$\mathbf{B} \begin{Bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{Bmatrix} = \mathbf{0} \qquad \mathbf{B}_{\text{modal}} = \mathbf{B} \begin{bmatrix} \mathbf{\Phi}^A & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}^B \end{bmatrix} \qquad \mathbf{B}_{\text{modal}} \begin{Bmatrix} \mathbf{q}^A \\ \mathbf{q}^B \end{Bmatrix} = \mathbf{0}$$

- Then eliminate some modal DOF using constraint equation to find $\mathbf{L}$ and multiply matrices to compute the unconstrained EOM. The "cms.m" routine implements all of this.

Matlab Implementation of Modal Substructuring

- The **cms.m** Matlab® routine by M.S. Allen implements all of the calculations needed for modal substructuring.
- See **cms_test_script_FE.m** for an example of how to call the function.

```
% Modal Substructuring (Component Mode Synthesis).  Combines substructures
% (add or subtract) defined by modal parameters or system matrices M,C,K
% according to the user supplied constraints and returns the modal
% parameters for the built-up system.
%
% [wn_mod,zt_mod,phi_mod,subsys_ind,all_data] = cms(subsys,carray,sgn_vec);
%
% INPUTS:
%   subsys - N_sub dimensional array with the following fields for each
%   substructure:  (i.e) subsys(1).wn = wn vector for subsystem #1.
%       wn - vector of natural frequencies for the un-modified structure.
%       zt - (optional) vector of modal damping ratios - set to zero
%           if not supplied.
%       phi - matrix of mass normalized mode vectors where each column
%           corresponds to a natural frequency in wn.
%       mmass - (optional) vector of modal masses - set to one if not
%           supplied.  (Set these to zero to add residual flexibility modes.)
%       names - (optional) vector of node names for subsystem.  These are
%           concatenated into "namespa" if supplied.
%       DOF - (optional, in place of "names") - numeric vector giving the
%           node/direction for each point. e.g. 101.1=Node 101, x-dir,
%           386.6=Node 386, z-rotation. These are concatenated into
%           "DOFpa" if supplied.
```


Matlab Implementation (continued)

```
%   OR
%       M,K - mass and stiffness matrices
%       C - (optional) damping matrix
%       names - (optional) vector of node names for subsystem.  These are
%           concatenated into a vector of names for phi_mod if supplied.
%
%   carray - 3D array containing the constraints between subsystems.  Each
%       constraint is a 3-column matrix with the following columns:
%       carray(:,:,C_num) = [a1, pt1, subsys1;
%                           a2, pt2, subsys2;
%                           ...];
%       where a1 are the coefficients in an equation of the form:
%           a1*y1+a2*y2+...=0,
%       subsys1 is the index of the subsystem being joined (y1 is on
%           subsys1),
%       and pt1 is the index of the node in subsys1 that is being joined.
%       For example:  To assign node 3 on subsystem 2 to have the same
%       displacement as node 4 on subsystem 1, the following matrix would
%       be used:  carray = [1, 3, 2; -1, 4, 1]
%           NOTE:  This assumes the coordinate systems are identical for
%           each component - use care if this is not the case!!
%       ##### See "mkconstr" or "mkmodalconstr" for alternative #####
%
%   sgn_vec - [optional] vector of length N_sub of +1 or -1 values if each
%       substructure is being added or subtracted respectively to the
%       master system.  All substructures are added if this is not
%       supplied.
%
...
```

Simplifying Input for “cms” with “mkconstr.m”

```
% Make constraints for cms.m or ritzscomb.m - creates constraint arrays to
% join pairsof subsystems given the subsystem index and the names of the
% nodes to be joined.
%
%   carray = mkconstr(subsys,s1,s2,nodes,s1,s2,nodes,...)
%
% subsys is a cell array containing the following for the kth subsystem:
%   subsys(k).wn = vector of nat freqs
%   subsys(k).zt = vector of damping ratos (optional)
%   subsys(k).phi = mode shape matrix (No x N where No = number of outputs,
%       N = number of modes)
%   subsys(k).names = cell array of names of each of the nodes (length No)
%
%   The result is:
%   carray = [1, ns1, s1; -1, ns2, s2]; etc... as needed for cms.m
%
% EXAMPLE:  Join two subsystems so that (y1)1=(y1)2 and (z1)1=(z1)2 displacement
%   subsys(1).phi = [3xN matrix]
%   subsys(1).names = {'x1','y1','z1');
%   subsys(2).phi = [4xN matrix]
%   subsys(2).names = {'x3','x1','y1','z1');
%   carray = mkconstr(subsys,1,2,{'y1','z1'})
%       returns the constraint array needed to enforce equal motion of
%       those two nodes between subsytems 1 and 2.
%       - One could then use [...] = ritzscomb(subsys,carray,[1,1]) to
%       compute the modes of the coupled system.
%
% Alternate Input: Provide DOF vectors saying which nodes to match.
%   nodes=[DOF1,DOF2]
%   (example) nodes=[101.1, 201.1;
%                    101.2, 201.2];
%   This constrains DOF 101.1 to DOF 201.1 and DOF 101.2 to DOF 201.2
%
```

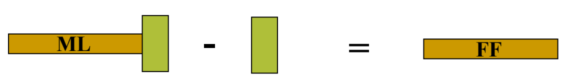

Solution to Exercise #2 using “cms.m”

```
% Modal substructuring solution:
subsys(1).wn=fn_A*2*pi;
subsys(1).zt=zt_A;
subsys(1).phi=Phi_A;

subsys(2).wn=fn_B*2*pi;
subsys(2).zt=zt_B;
subsys(2).phi=Phi_B;

% Define array that enforces the constraint x2=x3 => 1*x2-1*x3=0;
carray=[1,2,1; -1,1,2]; % the interpretation is as follows:
    % factor,coord_number,subsys_number
    % where the factors are (1) and (-1) and we are connecting coord. 2 on
    % subsystem 1 to coord 1 on subsystem 2.
[wn_MS,zt_MS,phi_MS] = cms(subsys,carray,[1,1])
    % Compare those values with
    fn_MS=wn_MS/2/pi

res=(phi_MS(4,:).*phi_MS(1,:)).'; % modal residues used to create FRFs for input
    at 1 and output at 4

H41MS=cl_model(wn_MS,zt_MS,res,ws,'A');

figure(3)
semilogy(ws/2/pi,abs(H41C),ws/2/pi,abs(H41MS),'.--'); grid on;
xlabel('\bfFrequency (Hz)'); ylabel('\bfFRF m/N');
title('\bfAssembly C: Modal Substructuring'); legend('H_{41}^C','H_{41}^C MS');
```

Additional Examples

Structural Modification: Example with ritzsmod.m

- Steel fixture (4.6”x1”x1” bar), treated as a mass and inertia added to the end of a 1”x0.75”x12” steel beam.

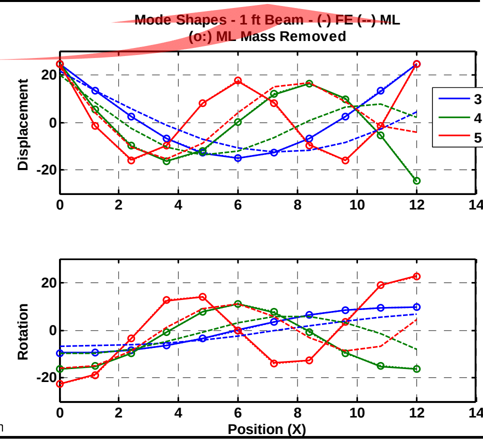


- Simulated in Salinas using 40 beam elements.
  - Weight of Beam: 2.5 lb
  - Weight of Fixture: 1.3 lb
- “ritzsmod.m” used (via “ritzsmod_test_script_FE.m”) used to remove mass and inertia.

Structural Modification Example

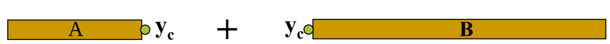
Natural Frequencies (Hz) and Errors (%)						
Mode #	ML (FE)	FF (FE)	FF (Ritz Method)	E (%) Ritz	A FF (Sensitivity)	E (%) Sensitivity
3	703.766	1086.19	1085.32	*-0.08%*	835.54	*-23.1%*
4	1669.55	2988.13	2986	*-0.07%*	2055.7	*-31.2%*
5	3356.85	5845.21	5837.76	*-0.13%*	3673.9	*-37.1%*
6	6128.5	9639.3	17705.6	*83.68%*	6367	*-33.9%*

- Natural Frequencies and Mode Shapes from full (40 node) FE model compared with those estimated by “ritzsmod” using 6 modes (all modes below 6400 Hz).
- Errors in all but the 6th (last) mode are tiny.
- Mode shapes are quite accurate
- Sensitivity estimate of natural frequencies has large errors (30%) because the mass addition is not small.

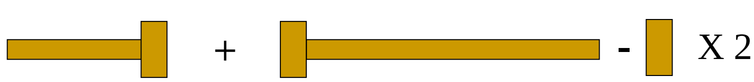


Joining Structures: Example Problem

- Ritz Method used to combine models for 12” and 24” beams using modes for each substructure up to 6400 Hz .
  - Case 1: Free-Free (FF) modes – modal parameters of each FF beam used to predict combined system response:



  - Case 2: Mass-Loaded (ML) modes – parameters of modes with 4.6in end blocks used for each beam.



Method

- Salinas models of 12” and 24” beams created with and without end blocks.
  - Each beam meshed with 0.3 inch long elements.
  - Solution of 12” and 24” beams joined was also found.
- The modal model for the case of Mass-Loaded modes was corrected using the “ritzsmod” script in order to remove the effects of the end masses. The modified modal models were then combined to form the 36-in beam.

Correcting for End Mass and Inertia

	Natural Frequencies (Hz)								
Mode #	**A ML**	**A ML-corr**	**B ML**	**B ML-corr**	**A FF**	**B FF**	**E (%) ML**	**E (%) ML**
3	703.766	1085.32	213.845	271.945	1086.19	272.055	-0.08%	-0.04%
4	1669.55	2986	598.158	749.272	2988.13	749.554	-0.07%	-0.04%
5	3356.85	5837.76	1122.28	1468.17	5845.21	1468.62	-0.13%	-0.03%
6	6128.5	17705.6	1769.53	2425.59		2426.24		-0.03%
7			2646.01	3620.69		3622		-0.04%
8			3796	5053.16		5055.21		-0.04%
9			5205.86	13059.8				

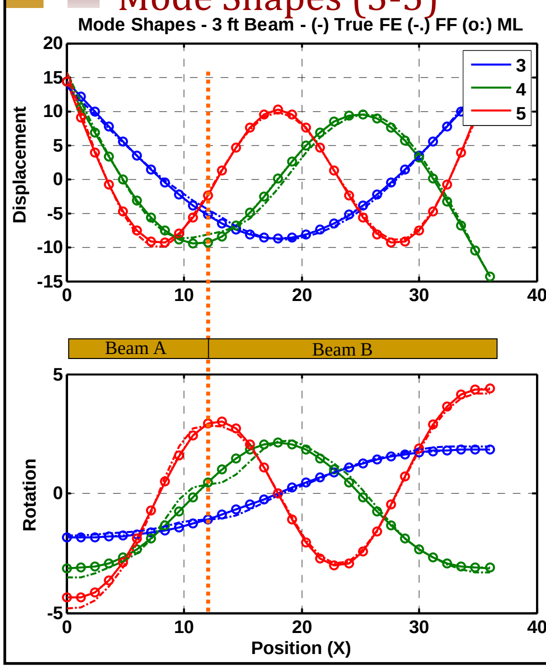
- Chart compares Mass-Loaded natural frequencies with those obtained after analytically subtracting off the mass using “ritzsmod.m” with the Free-Free natural frequencies for systems A and B.
- The natural frequencies predicted match the FF natural frequencies remarkably well.

Natural Frequencies for 36-in Beam

	Natural Frequencies (Hz)				
Mode #	**FEA**	**FF**	**ML**	**E (%) FF**	**E(%) ML**
3	121.0	128.0	121.0	5.8%	0.00%
4	333.3	364.0	333.3	9.2%	-0.02%
5	653.3	658.8	653.3	0.8%	0.00%
6	1079.7	1130.8	1079.7	4.7%	0.00%
7	1612.4	1781.8	1612.4	10.5%	0.00%
8	2251.3	2282.0	2252.7	1.4%	0.06%
9	2996.3	3176.3	3003.1	6.0%	0.23%
10	3847.2	4396.0	3855.6	14.3%	0.22%
11	4803.7	4934.3	4836.4	2.7%	0.68%

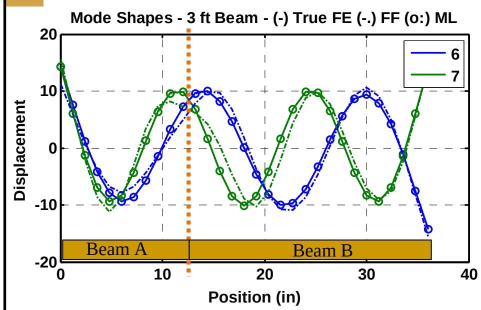
- Table shows natural frequencies of 36-in beam (1) predicted by FEA, (2) predicted using the Free-Free modes of the 12” and 24” beams and (3) predicted using Mass-Loaded modes of the 12” and 24” beams.
- Free-Free modes have errors ranging from 0% to 15%, while the Mass-Loaded modes show errors less than 1%.
- Error in natural frequency is largest for modes (of 36-in beam) that have large shear (derivative of rotation) at the connection point. (see mode shapes on next slide)

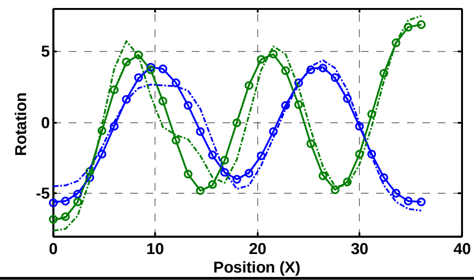
Mode Shapes (3-5)



- Mass loaded mode vectors match the analytical mode vectors more closely, especially the rotation component at the connection.
- Note: shear and moment in beam proportional to d^2u/dx^2 and d^3u/dx^3.
  - Free-Free modes all have zero shear and moment forces at the free end, so they cannot represent the combined structure well near the connection.

Mode Shapes (6-7)





- Shapes of modes 6 and 7 for the 36” beam.
- The FF modes are clearly not a very good basis for these modes.

Component Mode Synthesis

- This is only one of a few modal bases that have been used in analytical substructuring.
  - Craig-Bampton Method [1]:
    - Fixed interface modes + Constraint Modes
    - Efficient basis substructures that are connected rigidly
    - Rule of thumb: Include modes out to 1.5 to 2.0 times the frequency range of interest.
  - Method of MacNeal (variations by Rubin, etc…)
    - Free-interface modes + (variations on) Residual Flexibility
    - Has been found to give good results when subcomponents are weakly coupled (e.g. mounted on vibration isolation mounts).
  - Mass-loaded interface modes (as used in the previous example)
    - Provide a compromise between free-free and fixed-interface modes

[1] R. R. J. Craig, *Structural Dynamics*. New York: John Wiley and Sons, 1981